An International Peer Reviewed Journal of Physical Science

Acta Ciencia Indica, Vol. XLVII-P, No. 1 to 4, 19 (2021)

ROLE OF MAXIMALLY ENTANGLED STATES (MES) IN QUANTUM COMPUTING

BALWANT SINGH RAJPUT

Kumaun University, Nainital (Uttrakhand)
Present Address: I-11, Gamma-2, Greater Noida (UP)
Email: bsrajp@gmail.com

RECEIVED : 28 March, 2021

The entanglement has been explored as one of the key resources required for quantum computation, the functional dependence of the entanglement measures on spin correlation functions has been established, correspondence between evolution of MES of two-qubit system and representation of SU(2) group has been worked out in the new eigen basis (Rajput-Singh Eigen basis)

INTRODUCTION

Richard Feynman [1] examined the role quantum mechanics can play in the development of future computer hardware and demonstrated that time evolution of an arbitrary quantum state is intrinsically more powerful computationally than the evolution of logical classical state. Since then, quantum computing has attracted wide attention and soon became the hot topic of research. Quantum Computer (QC) [2, 3, 4, 5] is quantum information processing [6, 7, 8]. It is relatively new discipline and not yet completely understood. However, it provides an excellent introduction to many of key ideas. Measurement and manipulation of entangled state of many particles system becomes a far reaching consequence of quantum information processing. The physically allowed degree of entanglement [9] and mixture is a timely issue given that the entangled mixed states could be advantageous for certain quantum information situation. The simplest non-trivial multi-particle system that can be investigated theoretically, as well as experimentally, consists of two *Q*-bits which display many of the paradoxical features of quantum mechanics such as superposition and entanglement. Basis of entanglement is the correlation [10] that can exist between *Q*-bits. From physical point of view, entanglement is still little understood. What makes it too powerful is the fact that since quantum states exist as superposition, these correlations exist in superposition as well and when superposition is destroyed, the proper correlation is somehow communicated between the *Q*-bits. It is this communication that is the crux of entanglement. Entanglement is one of the key resources required for quantum computation and hence the experimental creation and measurement of entangled states is of crucial importance for various physical implementations of quantum computers. Quantum entanglement was already pointed out by Schrodinger to be a crucial element of quantum mechanics but the research has been refocused on it in the last fifteen years because the fields of quantum information theory, quantum computers, universal quantum computing network, teleportation [11], dense coding [12], geometric quantum computation [13] and quantum cryptography [14] are being developed rather quickly. By quantum entanglement we mean quantum correlation among the distinct subsystems of the entire composite system. For such correlated quantum systems, it is not possible to specify the quantum state of any subsystem independently of the remaining subsystems.The generation of quantum entanglement among spatially separated particles requires non-local interactions through which the quantum correlations are dynamically created but our present knowledge of quantum entanglement is not at all satisfactory [9]. However, the functional dependence of the entanglement measures like concurrence [15,16], i-concurrence [17] and 3-tangle [18] on spin- correlation functions [19] have been established. Protection of quantum states of open system from decoherence is essential for robust quantum information processing and quantum control in quantum computers.

In the present paper, entanglement has been explored as one of the key resources required for quantum computation, the functional dependence of the entanglement measures on spin correlation functions has been established and there presentation of SU(2) group has been worked out in the new eigen basis (Rajput-Singh eigen basis) [20-25]. It has been shown that the degree of entanglement for a two-qubit state depends on the extent of fractionalization of its density matrix and that the entanglement is completely a quantum phenomenon without any classical analogue. A reliable measure of entanglement of two-qubit states has also been expressed in terms of concurrence [15,16] and it has been shown that in a free two-qubit system the states with both combinations of parallel spins (*i.e.* states with maximum Hamming spread) are definitely maximally entangled states (MES) while among the states with minimum Hamming spread, those with both anti-parallel combinations are MES and those with one combination of parallel spins and other with anti-parallel spins are not entangled at all. Necessary and sufficient conditions for the general two-qubit state to be maximally entangled state have been obtained and the conditions for this state to be non-entangled (*i.e.* separable) and to be partially entangled respectively, have been derived.

THEORETICAL BASIS OF QUANTUM COMPUTING

At quantum level an electron can be in a superposition of many different energy states which is not possible classically. Similarly, any Physical system is described by quantum state

$$|\psi> = \sum_i C_i |\emptyset_i>$$

This is a linear superposition of basis states $|\emptyset_i>$. Such a state is a Coherent State**.** This superposition is destroyed on interaction of system with its environment, *i.e.* it becomes decoherent. $|C_i|^2$gives the probability of $|\psi>$ collapsing in to state $|\emptyset_i>$ as it decoheres.

Electron-spin is a two state system with elements $|\uparrow>$ corresponding to spin-up and $|\downarrow>$ corresponding to spin- down. A state of this system may be written as

$$|\psi >= a\,|\uparrow> +b|\downarrow>$$

As long as system maintains coherence, it cannot be said to be in either spin-up or spin-down. When it decoheres, it can be in either of these states. Such a simple two- state quantum system is the basic unit of quantum computation: quantum-bit (*Q*-bit) where we rename two states as 0-state, and 1-state. Smallest unit of information stored in a two-state quantum computer is called a *Q*-bit. If there is a system of *m* *Q*-bits, it can represent 2^m states at the same time.

Q-bit is simply a two-level system with generic state as:

$$|\psi> = a|0> +b|1>,$$

A two-dimensional complex vector, where $a\ and\ b$ are complex coefficients specifying the probability amplitudes of corresponding states such that

$$|a|^2 + |b|^2 = 1$$

Q-bit individual is defined by a string of Q-bits.

$$|\widetilde{0} > = |00 \ldots\ldots\ldots\ldots >$$

An operator on a Hilbert space describes how one eigen state is changed in to other.

$$|\emptyset > = \widehat{U}\,|\psi>$$

For instance,

Let $$|\psi\ > = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} = \frac{2}{\sqrt{5}}\ |\uparrow> + \frac{1}{\sqrt{5}}\ |\downarrow>$$

and an operator represented by matrix

$$\widehat{U} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

Then we have

$$\widehat{O}|\psi >= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}$$

$$\Rightarrow \psi > = \frac{3}{\sqrt{10}}\uparrow> + \frac{1}{\sqrt{10}}|\downarrow>$$

$\Rightarrow$ Amplitude of $|\uparrow>$ has increased while that of $|\downarrow>$ has decreased.

Thus a quantum operator is a *q*-gate and represented by a square matrix.

State of a *Q*-bit can be changed by the operation with a quantum gate which derives the individuals towards better solution (eventually towards a single state). A quantum gate is a reversible gate and can be represented as a unitary matrix U acting on a *Q*-bit basis state. *Q*-gates operating on just two bits at a time are sufficient to construct a general quantum circuit (based on Lie-Group theory).Thus quantum operator may be made to work as NOT- gate; controlled NOT gate (C-NOT); Rotation-gate; Hadmard-gate etc.

Quantum Computation (QC) can be defined as representing the problem to be solved in the language of quantum states and producing operators that derive the system to a final state such that when system is observed there is high probability of finding a solution. QC consists of state preparation; useful time evolution of quantum system; and measurement of the system to obtain information. Upon measurement system will collapse to a single basis state. Object of QC is to ensure that measured basis state is with high probability.

ENTANGLEMENT

Basis of entanglement is the correlation that can exist between *Q*-bits. These correlations exist in superposition as well and when superposition is destroyed, the proper correlation is somehow communicated between the *Q*-bits. It is this communication that is the crux of entanglement.

Mathematically, it is described using density matrix formulation; Density matrix of state $|\psi>$ is given by:

$$\rho_{\psi} = |\psi><\psi|$$

The state for which density matrix cannot be factorized is said to be entangled while those with fully factorized density matrix are not entangled at all. For instance, let us consider a two-qubit state as:

$$|\psi> \frac{1}{\sqrt{2}}[|00> + |11>]$$

It appears in matrix form as:

$$|\psi> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

where ‘1’ denotes the presence of the corresponding eigen state in the superposition and ‘0’ denotes its absence, i.e. ‘1’ for $|00>$ and $|11>$and ‘0’ for $|01>$ and $|10>$. This quantum state is the superposition of only the states $|00>$ and $|11>$ which have maximum Hamming spread between two-qubits. For this state we have the density matrix

$$\rho_{\psi} = |\psi><\psi| = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{bmatrix}$$

This cannot be factorized at all and the state $|\psi>$ is maximally entangled (MES).

Let us now consider the following quantum state as superposition of qubits $|00>$ and $|01>$ which have minimum Hamming spread;

$$|\varepsilon>=\frac{1}{\sqrt{2}}|00>+\frac{1}{\sqrt{2}}|01>$$

or $$|\varepsilon>=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

Its density matrix is:

$$\rho_\varepsilon=|\varepsilon><\varepsilon|=\frac{1}{2}\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0\\0&0\end{bmatrix}\otimes\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

which is completely factorized. This state is not entangled at all.

Another quantum state as superposition of Q-bits with least Hamming spread may be written as:

$$|\epsilon>=\frac{1}{\sqrt{2}}|01>+\frac{1}{\sqrt{2}}|11>=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

with density matrix

$$\rho_\epsilon=|\epsilon><\epsilon|=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

which is fully factorized and hence this state is not entangled at all.

On the other hand the quantum state as superposition of Q-bits $|00>, |01>$ and $|11>$ may be written as

$$|\zeta>=\frac{1}{\sqrt{3}}|00>+\frac{1}{\sqrt{3}}|01>+\frac{1}{\sqrt{3}}|11>=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$$

Its density matrix is:

$$\rho_\zeta=|\zeta><\zeta|=\frac{1}{3}\begin{bmatrix}1&1&0&1\\1&1&0&1\\0&0&0&0\\1&1&0&1\end{bmatrix}$$

This can be only partially factorized as

$$\rho_\zeta=\frac{1}{3}\left[\begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&1\end{pmatrix}+\begin{pmatrix}1&1&0&1\\1&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\right]$$

and hence the state is partially entangled. Thus the degree of entanglement for a two Q-bits state depends on the extent of fractionalization of its density matrix and the entanglement is completely quantum phenomena without classical analogue.

It may readily be shown that the density matrix for the following two Q-bit states (Bell States) cannot be factorized at all;

$$|\phi_1 > = -\frac{i}{\sqrt{2}}(|00 > -| 11 >);$$

$$|\phi_2 > = \frac{1}{\sqrt{2}}(|00 > +| 11 >);$$

$$|\phi_3 > = -\frac{i}{\sqrt{2}}(|01 > +| 10 >);$$

$$|\emptyset_4 > = \frac{1}{\sqrt{2}}(|01 > -| 10 >).$$

and hence all these states are maximally entangled states (MES). The matrices of these states satisfy the condition:

$$\sum_{\mu=1}^{4} \rho_{\emptyset_\mu} = 1$$

or

$$\sum_{\mu=1}^{4} | \phi_\mu >< \phi_\mu | = 1$$

These states also satisfy the condition

$$< \phi_\mu | \phi_\upsilon > = \delta_{\mu\nu}$$

These equations show that the Bell's states constitute the orthonormal complete set and hence form the eigen-basis (magic basis) of the space of two level Q-bits. These states are maximally entangled states (MES).

For pure state $| \psi >$ any two-qubit state may be written in magic basis as:

$$|\psi > = \sum_{k=1}^{4} b_k |\phi_k >$$

with its concurrence defined as

$$| C(|\psi >) = | \sum_{k=1}^{4} b_k^2 |$$

If the concurrence $C(|\psi >) = 1$, the state is maximally entangled while for $C(| \psi >) = 0$, the state $| \psi >$ is not entangled at all.

For $0 < C(| \psi >) < 1$,

the state $| \psi >$ is partially entangled.

The concurrence of a state is as reliable measure of degree of entanglement as the extent of factorization of its density matrix while Hamming spread of a two *Q*-bits state is not that reliable measure of the entanglement since the states $| \varepsilon >$ and $| \epsilon >$ with minimum Hamming spread and zero concurrence are not entangled at all (*i.e.* completely separable) and the states $| \emptyset_3 >$ and $| \emptyset_4 >$ with minimum Hamming spread but concurrence unity, are maximally entangled states (MES).

In terms of Z-components of spins of two electrons, the states $|\emptyset_3>$ and $|\emptyset_4>$of magic bases with minimum Hamming spread, may be written as:

$$|\emptyset_3> = |\uparrow\downarrow> + |\downarrow\uparrow> \text{ and } |\emptyset_4> = |\uparrow\downarrow> - |\downarrow\uparrow>$$

This consists of Q-bits with anti-parallel spins. On the other hand, the states $|\varepsilon>$ and $|\in>$ with minimum Hamming spreads may be written as:

$$|\varepsilon> = |\uparrow\uparrow> + |\uparrow\downarrow> \text{ and } |\in> = |\uparrow\downarrow> + |\downarrow\downarrow>$$

This is with one combination of parallel spins and other of anti-parallel spins. In states $|\emptyset_1>$ and $|\emptyset_2>$ both combinations are with parallel spins. Thus in free two-qubit system the states with combinations of parallel spins (*i.e.* states with maximum Hamming separation) are definitely MES while among the states with minimum Hamming spread, those with anti-parallel spins are MES and those with one combination of parallel spins and other with anti-parallel spins are not entangled at all. Therefore, various qubits of two-qubit states may be written as follows in magic basis;

$$|00> = |\uparrow\uparrow> = \frac{i}{\sqrt{2}}|\emptyset_1> + \frac{1}{\sqrt{2}}|\emptyset_2>,$$

$$|01> = |\uparrow\downarrow> = \frac{i}{\sqrt{2}}|\emptyset_3> + \frac{1}{\sqrt{2}}|\emptyset_4>,$$

$$|10> = |\downarrow\uparrow> = \frac{i}{\sqrt{2}}|\emptyset_3> - \frac{1}{\sqrt{2}}|\emptyset_4>,$$

$$|11> = |\downarrow\downarrow> = -\frac{i}{\sqrt{2}}|\emptyset_1> + \frac{1}{\sqrt{2}}|\emptyset_2>$$

NECESSARY AND SUFFICIENT CONDITIONS FOR A TWO-QUBIT STATE TO BE MES

A general two-qubit state may be written as:

$$|\Psi> = \frac{1}{\sqrt{\gamma}}[\,a|00> + b|01> + c|10> + d|11>]$$

$$= \frac{1}{\sqrt{\gamma}}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

where $\gamma = |a|^2 + |b|^2 + |c|^2 + |d|^2$

This state may be written as:

$$|\Psi> = \frac{1}{\sqrt{(2\gamma)}}[\,i(a-d)|\emptyset_1> + (a+d)\,|\emptyset_2> + i(b+c)|\emptyset_3> + (b-c)\,|\emptyset_4>]$$

and its concurrence becomes

$$C(|\Psi>) = \frac{2}{\gamma}|ad - bc|$$

Thus, for non-entangled state (*i.e.* separable state), we have:

$$ad = bc$$

and for partially entangled states,

$$0 < \frac{2|ad - bc|}{\gamma} < 1$$

For MES, we have:

$$2|ad - bc| = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

or $$|(a \mp d^*)|^2 + |(b \pm c^*)|^2 = 0$$

This can be true either for;

$$d = a^* \; and \; c = -b^*$$

or for; $d = -a^* \; and \; c = b^*$

These are the necessary conditions for the state $|\Psi>$ of equation to be maximally entangled. Thus, we get the following two sets of MES:

$$|\Psi_1> = \frac{1}{\sqrt{(2[|a|^2 + |b|^2\}}} [\, a|00> \; + b|01> - b^*|10> \; + a^*|11>]$$

and $$|\Psi_2> = \frac{1}{\sqrt{(2[|a|^2+|b|^2\}}} [\, a|00> \; + b|01> + b^*|10> \; - a^*|11>]$$

Bell states (*i.e.* magic bases) may readily be obtained from the state $|\Psi_1>$ on substituting:

$(a = 1, b = 0); (a = -i, b = 0); (a = 0, b = 1); and \; (a = 0, b = -i)$

For these sets of values of a and b, the state $|\Psi_2>$gives $|\emptyset_1>$ and $|\emptyset_4>$ with phase rotated by $\frac{\pi}{2}$ and $|\emptyset_2>$ and $|\emptyset_3>$ with phase rotated by $-\frac{\pi}{2}$.

Other maximally entangled two-qubit states which form the orthonormal complete set (i.e. eigen basis) may be obtained as follows by putting $a = \pm 1$ and $b = 1$ in state $|\Psi_2>$ of equation and $a = 1, b = \pm 1$ in state $|\Psi_1>$;

$$|\psi_1> = \frac{1}{2}[-|00> \; +|01> +|10> \; +|11>],$$

$$|\psi_2> = \frac{1}{2}[\,|00> \; -|01> +|10> \; +|11>],$$

$$|\psi_3> = \frac{1}{2}[\,|00> \; +|01> -|10> \; +|11>], \quad \ldots (A)$$

$$|\psi_4> = \frac{1}{2}[\,|00> \; +|01> +|10> \; -|11>]$$

with their density matrices respectively given by:

$$\rho_{\psi_1} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix},$$

$$\rho_{\psi_2} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

$$\rho_{\psi_3} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\rho_{\psi_4} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

None of which can be factorized at all. The concurrence for each of these states is unity and these states constitute the orthonormal set since

$$< \psi_\mu | \psi_\nu > = \delta_{\mu\nu} \text{ and } \sum_{\mu=1}^{4} |\psi_\nu >< \psi_\mu| = I$$

Other six MES obtained from $|\Psi_1>$ and $|\Psi_2>$ by substituting $(a = 1, b = \pm i)$ and $(a = i, b = \pm 1)$ respectively, do not constitute complete set (*i.e.* do not form eigen bases).

States given by equation (A) also constitute the eigen basis (different from magic basis) of the space of two- qubit system. This is the new eigen basis, being introduced for the first time, and to differentiate it from the already known Bell's basis, let us call it **Singh-Rajput basis**, for its possible use in future in the literature. The MES constructed in the form given by eqns. (A) may be correspondingly called **Singh-Rajput states [20-25].** In terms of these states, all the qubits of two-qubit system may be obtained as

$$|00> = \frac{1}{2}\,[\,|\psi_2> + |\psi_3> + |\psi_4> - |\psi_1>],$$

$$|01> = \frac{1}{2}\,[\,|\psi_1> + |\psi_3> + |\psi_4> - |\psi_2>],$$

$$|10> = \frac{1}{2}\,[\psi_1> + |\psi_2> + |\psi_4> - |\psi_3>],$$

$$|11> = \frac{1}{2}\,[\,|\psi_1> + |\psi_2> + |\psi_3> - |\psi_4>]$$

Bell states may be constructed as follows in this new basis;

$$|\phi_1> = \frac{-i}{\sqrt{2}}[\,|\psi_4> - |\psi_1>];\ |\phi_2> = \frac{1}{\sqrt{2}}[\,|\psi_2> + |\psi_3>];$$

$$|\phi_3> = \frac{-i}{\sqrt{2}}[|\psi_4> + |\psi_1>];\ \ |\phi_2> = \frac{1}{\sqrt{2}}[\,|\psi_3> - |\psi_3>]$$

Concurrence of each of Bell states in this basis also is unity showing the invariance of concurrence in different bases. Condition for partial entanglement shows that if any coefficient of qubits in the state $|\Psi>$ is vanishing, then the state is necessarily partially entangled and its concurrence is $\frac{2}{3}$ if the sum of squares of moduli of non-zero coefficients is 3. For instance, let $b = 0$, and $|a|^2 + |c|^2 + |d|^2 = 3$, then the concurrence becomes $\frac{2}{3}$ when $a = \pm 1$, $c = \pm 1$ and $d = \pm 1$. It may be readily shown that all the states $\frac{1}{\sqrt{3}}[\pm|00> \pm|01> \pm|11>]$ are partially entangled with concurrence $= \frac{2}{3}$.

FUNCTIONAL DEPENDENCE OF ENTANGLEMENT ON SPIN OPERATORS OF QUBITS CONSTITUTING TWO-QUBIT STATES

The elements of density operator ρ of the state constituted by the qubits A *and* B can be expressed as follows in terms of spin matrices and the operators $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ associated with these qubits;

$$\rho_{11} = \frac{1}{4}[1 - <\sigma_A^z> - <\sigma_B^z> + <\sigma_A^z\sigma_B^z>] = [1 - T_r(\rho\sigma_A^z) - T_r(\rho\sigma_B^z) + T_r(\rho\sigma_A^z\sigma_B^z)]$$

$$\rho_{22} = \frac{1}{4}[1 - <\sigma_A^z> + <\sigma_B^z> - <\sigma_A^z\sigma_B^z>] = [1 - T_r(\rho\sigma_A^z) + T_r(\rho\sigma_B^z) - T_r(\rho\sigma_A^z\sigma_B^z)]$$

$$\rho_{33} = \frac{1}{4}[1 + <\sigma_A^z> - <\sigma_B^z> - <\sigma_A^z\sigma_B^z>] = [1 + T_r(\rho\sigma_A^z) - T_r(\rho\sigma_B^z) - T_r(\rho\sigma_A^z\sigma_B^z)]$$

$$\rho_{44} = \frac{1}{4}[1 + <\sigma_A^z> + <\sigma_B^z> + <\sigma_A^z\sigma_B^z>] = [1 + T_r(\rho\sigma_A^z) + T_r(\rho\sigma_B^z) + T_r(\rho\sigma_A^z\sigma_B^z)]$$

$$\rho_{23} = \rho_{23}^* = \frac{1}{4}[<\sigma_A^x\sigma_B^x> + <\sigma_A^y\sigma_B^y> + i\{<\sigma_A^y\sigma_B^x> - <\sigma_A^x\sigma_B^y>\}]$$

and all other elements vanishing. Thus the density matrix of the concerned state is

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}$$ with its eigen values given by

$$\alpha_1 = \rho_{11}, \alpha_2 = \rho_{44},$$

$$\alpha_3 = \frac{1}{2}[(\rho_{22} + \rho_{33}) + \sqrt{\{(\rho_{22} + \rho_{33})^2 - 4(\rho_{22}\rho_{33} - |\rho_{23}|^2)\}}]$$

$$= \frac{1}{4}\left[1 - <\sigma_A^z\sigma_B^z> + \sqrt{\begin{Bmatrix} <\sigma_A^z>^2 + <\sigma_B^z>^2 - 2<\sigma_A^z\sigma_B^z> \\ + \\ <\sigma_A^x\sigma_B^x>^2 + <\sigma_A^y\sigma_B^y>^2 + <\sigma_A^y\sigma_B^x>^2 + <\sigma_A^x\sigma_B^y>^2 \end{Bmatrix}}\right],$$

$$\alpha_4 = \frac{1}{2}[(\rho_{22}+\rho_{33}) - \sqrt{\{(\rho_{22}+\rho_{33})^2 - 4(\rho_{22}\rho_{33} - |\rho_{23}|^2)\}}]$$

$$= \frac{1}{4}\left[1 - <\sigma_A^z\sigma_B^z> - \sqrt{\begin{Bmatrix} <\sigma_A^z>^2 + <\sigma_B^z>^2 - 2<\sigma_A^z\sigma_B^z> \\ + \\ <\sigma_A^x\sigma_B^x>^2 + <\sigma_A^y\sigma_B^y>^2 + <\sigma_A^y\sigma_B^x>^2 + <\sigma_A^x\sigma_B^y>^2 \end{Bmatrix}}\right]$$

which give $\sum_{j=1}^{4}\alpha_j = \rho_{11}+\rho_{22}+\rho_{33}+\rho_{44} = T_r\rho = 1$

For $\alpha_1 = \alpha_2$, we have

$$<\sigma_A^z> = -<\sigma_B^z>$$

and then $\alpha_3 > \alpha_1$.

For this choice of eigen values of density matrix, the necessary and sufficient condition for the concerned state to be entangled is

$$\alpha_3 > \alpha_1 + \alpha_2 + \alpha_4$$

while the state is separable (not entangled at all) if

$$\alpha_3 = \alpha_1 + \alpha_2 + \alpha_4$$

Thus we get the following necessary and sufficient conditions for entanglement;

$$<\sigma_A^z><\sigma_B^z> > <\sigma_A^z\sigma_B^z>$$

and$[\ (1-<\sigma_A^z\sigma_B^z>)^2 - (<\sigma_A^z> + <\sigma_B^z>)^2] \ < [(<\sigma_A^x\sigma_B^x> + <\sigma_A^y\sigma_B^y>)^2 + (<\sigma_A^y\sigma_B^x> - <\sigma_A^x\sigma_B^y>)^2]$

In case the state exhibits spin- flip symmetry, we have

$$<\sigma_A^z> = 0;\ <\sigma_B^z> = 0\ and\ \ <\sigma_A^y\sigma_B^x> = <\sigma_A^x\sigma_B^y>$$

Then conditions for entanglement reduce to

$$<\sigma_A^z\sigma_B^z> < 0$$

$$and\ \ (1-<\sigma_A^z\sigma_B^z>) < (<\sigma_A^x\sigma_B^x> + <\sigma_A^y\sigma_B^y>)$$

$$or\ \ (<\sigma_A^x\sigma_B^x> + <\sigma_A^y\sigma_B^y> + <\sigma_A^z\sigma_B^z>)\ > 1$$

These are necessary and sufficient conditions of entanglement of a state in terms of correlation of spin components of its constituent qubits. The first condition shows that if the state of constituent qubits A and B with $<\sigma_A^z> = <\sigma_B^z> = 0$, is entangled then the z-components of the spins must be correlated antiferromagnetically.

For maximal entanglement, the eigen values of density matrix satisfy the following condition

$$\alpha_3 - \alpha_1 - \alpha_2 - \alpha_4 = 1$$

or

$$\alpha_3 - \alpha_4 = 1 + \alpha_1 + \alpha_2$$

which gives

$$(<\sigma_A^x\sigma_B^x> + <\sigma_A^y\sigma_B^y> + <\sigma_A^z\sigma_B^z>) = 2$$

Maximally entangled states satisfying this condition may find enormous applications to quantum communication and quantum computation techniques. Bell states which constitute magic basis as well as the new MES which constitute the new eigen basis for the space of two-qubit system, satisfy these conditions.

CORRESPONDENCE BETWEEN EVOLUTION OF MES OF TWO-QUBIT SYSTEM AND REPRESENTATIONS OF SU(2) GROUP

General MES for a two-qubit system for which $|a|^2 + |b|^2 = 1$; may also be written as follows

$$|\Psi_1> = |(\mathrm{a},\mathrm{b})>_+ = \frac{1}{\sqrt{2}}[\,\mathrm{a}|00> + b|01> - \mathrm{b}^*|10> + \mathrm{a}^*|11>]$$

and $$|\Psi_2> = |(a,b)>_- = \frac{1}{\sqrt{2}}[\,a|00> + b|01> + b^*|10> - a^*|11>]$$

$$|(a,b)>_\pm = \frac{1}{\sqrt{2}}[\,a|00> + b|01> \mp b^*|10> \pm a^*|11>]$$

$$= \frac{1}{\sqrt{2}}[\,a|\uparrow\uparrow> + b|\uparrow\downarrow> \mp b^*|\downarrow\uparrow> \pm a^*|\downarrow\downarrow>]$$

Let us substitute

$$a = cos\frac{\alpha}{2} - ik_z\,sin\frac{\alpha}{2}\;;\; b = -(k_y + ik_x)sin\frac{\alpha}{2}$$

where $\vec{k} = (k_x, k_y, k_z)$ is unit vector and $0 \le \alpha \le \pi$.

MES may then be written as

$$|\psi(\vec{k},\alpha)>_\pm = [cos\frac{\alpha}{2}(|00> \pm |11>) - ik_z sin\frac{\alpha}{2}\,(|00> \pm |11>)$$
$$-k_y sin\frac{\alpha}{2}(|01> \mp |10>) - ik_x sin\frac{\alpha}{2}(|01> \pm |10>)]$$

or in new eigen basis

$$|\psi(\vec{k},\alpha)>_+ = -\frac{ik_x}{\sqrt{2}}sin\frac{\alpha}{2}|\psi_1> + \frac{1}{\sqrt{2}}\left(cos\frac{\alpha}{2} - ik_z sin\frac{\alpha}{2} + k_y sin\frac{\alpha}{2}\right)|\psi_2>$$
$$+\frac{1}{\sqrt{2}}\left(cos\frac{\alpha}{2} - ik_z sin\frac{\alpha}{2} - k_y sin\frac{\alpha}{2}\right)|\psi_3> - \frac{ik_x}{\sqrt{2}}sin\frac{\alpha}{2}|\psi_4>$$

and $|\psi(\vec{k},\alpha)>_- = \frac{1}{\sqrt{2}}\left(-cos\frac{\alpha}{2} + ik_z sin\frac{\alpha}{2} - k_y sin\frac{\alpha}{2}\right)|\psi_1> + \frac{ik_x}{\sqrt{2}}sin\frac{\alpha}{2}|\psi_2>$

$$-\frac{ik_x}{\sqrt{2}}sin\frac{\alpha}{2}|\psi_3> + \frac{1}{\sqrt{2}}\left(cos\frac{\alpha}{2} - ik_z sin\frac{\alpha}{2} - k_y sin\frac{\alpha}{2}\right)|\psi_4>$$

Then we have

$$|\psi(\vec{k},\pi+\alpha)>_{+}=-\frac{ik_x}{\sqrt{2}}cos\frac{\alpha}{2}|\psi_1> +\frac{1}{\sqrt{2}}\left(-sin\frac{\alpha}{2}-ik_z cos\frac{\alpha}{2}+k_y cos\frac{\alpha}{2}\right)|\psi_2>$$
$$+\frac{1}{\sqrt{2}}\left(-sin\frac{\alpha}{2}-ik_z cos\frac{\alpha}{2}-k_y cos\frac{\alpha}{2}\right)|\psi_3>-\frac{ik_x}{\sqrt{2}}cos\frac{\alpha}{2}|\psi_4>$$

and $|\psi(-\vec{k},\pi+\alpha)>_{+}=\frac{ik_x}{\sqrt{2}}\cos\frac{\alpha}{2}|\psi_1> +\frac{1}{\sqrt{2}}\left(\sin\frac{\alpha}{2}+ik_z\cos\frac{\alpha}{2}-k_y\cos\frac{\alpha}{2}\right)|\psi_2>$

$$+\frac{1}{\sqrt{2}}\left(\sin\frac{\alpha}{2}+ik_z\cos\frac{\alpha}{2}+k_y\cos\frac{\alpha}{2}\right)|\psi_3>+\frac{ik_x}{\sqrt{2}}\cos\frac{\alpha}{2}|\psi_4>$$

These equations show that

$$|\psi(\vec{k},\pi+\alpha)>_{+}=-|\psi(-\vec{k},\pi+\alpha)>_{+}=e^{i\pi}|\psi(-\vec{k},\pi-\alpha)>_{+}$$

In the similar manner we have

$$|\psi(\vec{k},\pi+\alpha)>_{-}=-|\psi(-\vec{k},\pi-\alpha)>_{-}=\boldsymbol{e^{i\pi}}|\psi(-\vec{k},\pi-\alpha)>_{-}$$

These equations show that $(\vec{k},\pi+\alpha)$ *and* $(-\vec{k},\pi-\alpha)$ correspond to the same state with additional phase of π. This is just the case of the double valued representation of SO(3) group homomorphic on to group SU(2). It may be written as

$$D^{\frac{1}{2}}(\vec{k},\alpha)=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

for state $|(a,b)>_{+}$ and

$$D^{\frac{1}{2}}(\vec{k},\alpha)=\begin{pmatrix} a & b \\ b^* & -a^* \end{pmatrix}$$

for state $|(a,b)>_{-}$. Each of these matrices corresponds to a rotation $(\vec{k},\alpha)$) in a real space. Since $R(\vec{k},\pi+\alpha)$ and $R(-\vec{k},\pi-\alpha)$ are the same rotations, we have

$$D^{\frac{1}{2}}(\vec{k},\pi+\alpha)=-D^{-1/2}(-\vec{k},\pi-\alpha)$$

showing that there is one-one correspondence between the two-Qubit MES and the double valued representation of SO(3).Thus any MES of new eigenbasis can be evolved by a rotation from an initial MES $|(1,0)>_{+}$ and we may define a SO(3) sphere with diameter π filled by vectors $\alpha\vec{k}$. Thus a new MES corresponds to a point in the SO(3) sphere, an evolution of MES corresponds to a trajectory connecting two points, and the initial state $|(1,0)>_{+}$ locates at the centre of the SO(3) sphere.

Thus entanglement has been explored as one of the key resources required for quantum computation, the functional dependence of the entanglement measures on spin correlation functions has been established, correspondence between evolution of MES of two-qubit system and representation of SU(2) group has been worked out in the new Eigen basis (**Rajput-Singh** Eigen basis) [26-27].

REFERENCES

1. R.P. Feynman, *Int. Theor. Phys*., **26(21)**, 467-488 (1982).
2. D. Simon, SIAM *Journ. Comp*. **26(5)**, 1474-1483 (1997).
3. A. Ezkov, A. Nifanava and D. Ventura, Information Sciences **128**, 271-293 (2000).
4. Z.S. Wang, C. Wu, X.L. Feng, L.C. Kwek, C.H. Lai, C.H. Oh, and V. Vedral, *Phys. Rev.* **A76,** 044303-307 (2007).
5. Z.S, Wang, *Phys. Rev.* **A79,** 024304-308 (2009).
6. P.W. Shor, Proc. 35th Ann. Symp., Found of Computer Science, Los. Alamitos IEEE Comp. Press, 20-22 (1994).
7. L.K. Grover, Proc. 28th Ann. ACM. Symp. On Theory of Computing, Philadelphia, Pennsylvania, ACM Press, 212-221 (1996).
8. H.T. Tan, W.M. Zhang and G. Li, *Phys. Rev.* **A83,** 032102-108 (2011).
9. G. Benenti and G. Casati, *Phys. Rev***. E79,** 025201R-205R (2009).
10. A. Smirne, H.P. Breuer, J. Piilo, and B. Vacchini, *Phys. Rev.* **A84,** 062114-119 (2010).
11. S.S. Li, Y.Y. Nie, Z.H Hong, X.J. Yi and Y.B. Huang, Comm. *Theor. Phys.* **50,** 633-640 (2008).
12. Y.B. Huang, S.S. Li and Y.Y.Nie, *Int. Journ. Theor. Phys.* **48,** 95-100 (2009).
13. Z.S, Wang, *Phys. Rev*. **A79,** 024304-308 (2009).
14. W. Tittel, J. Bendel,, H. Zbinden and N. Gisin, *Phys. Rev. Lett.* **84,** 4737-4740 (2000).
15. S. Hill and W.K. Wooters, *Phys. Rev. Lett.* **78(26),** 5022-5025 (1997).
16. W.K. Wooters, *Phys. Rev. Lett***. 80(10),** 2245-2248 (1998).
17. P. Rungta, V. Buzek, C.M. Caves, M. Hillery and G. J. Millburn, *Phys. Rev*. **A64(4)**, 042315-320 (2001).
18. V. Coffman, J. Kundu and W.K. Wooters, *Phys. Rev*. **A61(5),** 052306 (2000).
19. Ulrich Glaser, Helmut Buttner and Holger Fehske, *Phys. Rev*. **A68,** 032318-326 (2003).
20. Manu P. Singh and B.S. Rajput, Maximally Entangled States of Two-Qubit *Systems Int. Journ. Theor. Phys*. **52,** 4237-4255 (2013).
21. Manu P. Singh and B.S. Rajput, Applications of Singh-Rajput MES in Recall operations of Quantum Associative Memory for a Two-Qubit System, *Int. Journ. Theor. Phys.* **54(10),** 3443-34460 (2015).
22. Manu P. Singh and B.S. Rajput, Processes of Quantum Associative Memory (QuAM) through New Maximally Entangled States (Singh-Rajput MES), *Int. Journ. Theor. Phys.* **55(2),** 124-140 (2016).
23. Manu P. Singh and B.S. Rajput, New Maximally Entangled States for Pattern Association Through Evolutionary Process in a Two-qubit System, *Int. Journ. Theor. Phys.* **56**, 1274-1285 (2017).
24. Manu P. Singh and B.S. Rajput, New maximally entangled states and Pattern classification, *Int. Journ. Theor. Phys.* **53(9),** 3226-3238 (2014).
25. Manu P. Singh and B.S. Rajput, Pattern Classification in two-qubit and three-qubit systems, *Euro. Phys. J. Plus* **129** (57), 1-13 (2014).
26. Manu P. Singh and B.S. Rajput, Pattern Classifications Using Grover's and Venchura's Algorithm in a Two- Qubit System, *Int. Journ. Theor. Phys.* **57**, 692-705 (2018).
27. Manu P. Singh and B.S. Rajput, New Maximally Entangled States for Pattern Association Through Evolutionary Process in a Two-qubit System, *Int. Journ. Theor. Phys.* **56** (4), 1274-1285 (2017).

❑